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Prime Gamma Rings with Centralizing and Commuting Generalized Jordan Derivations

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Abstract

Let M be a prime Γ-ring satisfying a certain assumption and D a nonzero derivation on M. Let $f: M \to M$ be a generalized Jordan derivation such that f is centralizing and commuting on a left ideal J of M. Then we prove that M is commutative.

Keywords: Prime Γ-ring, Centralizing and Commuting, Derivation, Jordan derivation, Generalized derivations, Generalized Jordan derivations

Introduction

The concept of a Γ-ring was first introduced by Nobusawa [13] and also shown that Γ-rings, more general than rings. Bernes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Bresar [2] studied centralizing mappings and derivations in prime rings. Kyuno [9] , Luh [10], [11], Hoque and Paul [5], [6] and others were obtained a large numbers of important basic properties of Γ-rings in various ways and determined some more remarkable results of Γ-rings. Ceven [3] studied on Jordan left derivations on completely prime Γ-rings. Mayne [12] have developed some remarkable result on prime rings with commuting and centralizing. Jaya Subba Reddy et.al [8] studied centralizing and commutating left generalized derivation on prime ring is commutative. Hoque and Paul [7] studied prime gamma rings with centralizing and commuting generalized derivations is a commutative. In this paper, following [7], we extended some results on prime gamma rings with centralizing and commuting generalized Jordan derivations.

Let M and Γ be additive abelian groups. If there exists a mapping $(x,\alpha,y)\to x\alpha y$ of $M\times\Gamma\times M\to M$, which satisfies the conditions
(i) $x\alpha y\in M$
(ii) $(x+y)\alpha z=x\alpha z+y\alpha z$, $x(\alpha+\beta)z=x\alpha z+x\beta z$, $x\alpha(y+z)=x\alpha y+x\alpha z$
(iii) $(x\alpha y)\beta z=x\alpha(y\beta z)$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$, then M is called a Γ-ring.
Every ring M is a Γ-ring with $M=\Gamma$. However a Γ-ring need not be a ring. Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U\subset U(U\Gamma M\subset U)$.
If U is both a left and a right ideal, then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2x=0$ implies $x=0$ for all $x\in M$. An ideal P_1 of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B\subseteq P_1$ implies $A\subseteq P_1$ or $B\subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U\subseteq P_2$ implies $U\subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b=(0)$ with $a,b\in M$, implies $a=0$ or $b=0$ and semiprime if $a\Gamma M\Gamma a=(0)$ with $a\in M$ implies $a=0$. Furthermore, M is said to be commutative Γ-ring if $x\alpha y=y\alpha x$ for all $x,y\in M$ and $\alpha\in\Gamma$. Moreover, the set $Z(M)=\{x\in M: x\alpha y=y\alpha x$ for all $y\in M$ and $\alpha\in\Gamma\}$ is called the centre of the Γ-ring M. If M is a Γ-ring, then $[x,y]_\alpha=x\alpha y-y\alpha x$ is known as the commutator of x and y with respect to α, where $x,y\in M$ and $\alpha\in\Gamma$. We make the basic commutator identities:
$[x\alpha y,z]_\beta=[x,z]_\beta\alpha y+x\alpha[y,z]_\beta$ and $[x,y\alpha z]_\beta=[x,y]_\beta\alpha z+y\alpha[x,z]_\beta$, for all $x,y\in M$ and $\alpha\in\Gamma$. We consider the following assumption:
$x\alpha y\beta z=x\beta y\alpha z$, for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. .... (A).
An additive mapping $D: M\to M$ is called a derivation if $D(x\alpha y)=D(x)\alpha y+x\alpha D(y)$ holds for all $x,y\in M$ and $\alpha\in\Gamma$. An additive mapping $D: M\to M$ is called a Jordan derivation if $D(x\alpha x)=D(x)\alpha x+x\alpha D(x)$ holds for all $x,y\in M$ and $\alpha\in\Gamma$. A mapping f is said to be commuting on a left ideal J of M if $[f(x),x]_\alpha=0$ for all $x\in J$ and $\alpha\in\Gamma$ and f is said to be centralizing if $[f(x),x]_\alpha\in Z(M)$ for all $x\in J$ and $\alpha\in\Gamma$. An additive mapping $f: M\to M$ is said to be a generalized derivation on M, if $f(x\alpha y)=f(x)\alpha y+x\alpha D(y)$ holds for all $x,y\in M$ and $\alpha\in\Gamma$, where D is a derivation on M. An additive mapping $f: M\to M$ is called a generalized Jordan derivation on M, if $f(x\alpha x)=f(x)\alpha x+x\alpha D(x)$ holds for all $x\in M$ and $\alpha\in\Gamma$, where D is a derivation on M.

Preliminaries and main results

We have to make some use of the following well-known results:

Remark 1: Let M be a prime Γ-ring. If $a\alpha b\in Z(M)$ with $0\neq a\in Z(M)$, then $b\in Z(M)$.

Remark 2: Let M be a prime Γ-ring and J a nonzero left ideal of M. If D is a nonzero derivation on M, then D is also a nonzero on J.

Remark 3: Let M be a prime Γ-ring and J a nonzero left ideal of M. If J is commutative, then M is also commutative.

Lemma 1: Suppose M is a prime Γ-ring satisfying the assumption (A) and $D: M \to M$ be a Jordan derivation. For an element $a \in M$, if $a\alpha D(x) = 0$, for all $x \in M$ and $\alpha \in \Gamma$, then either $a = 0$ or $D = 0$.

Proof: By our assumption, $a\alpha D(x) = 0$, for all $x \in M$, and $\alpha \in \Gamma$.
We replacing x by $x\beta x$ in above equation then, we get
$a\alpha D(x\beta x) = 0$

$a\alpha D(x)\beta x + a\alpha x\beta D(x) = 0$

$a\alpha x\beta D(x) = 0$, for all $x \in M$, and, $\alpha, \beta \in \Gamma$.

If D is nonzero, that is, if $D(x) \neq 0$, for some $x \in M$. Then by definition of prime Γ-ring, $a = 0$.

Lemma 2: Suppose M is a prime Γ-ring satisfying the assumption (A) and J a nonzero left ideal of M. If M has a derivation D which is zero on J, then D is zero on M.

Proof: By the hypothesis, $D(J) = 0$.

Replacing J by $M\Gamma J$ in above equation then, we get

$D(M\Gamma J) = 0$

$D(M)\Gamma J + M\Gamma D(J) = 0$

$D(M)\Gamma J = 0$.

By lemma 1, D must be zero, since J is nonzero.

Lemma 3 [7]: Suppose M is a prime Γ-ring satisfying the assumption (A) and J a nonzero left ideal of M. If J is commutative on M, then M is commutative.

Lemma 4: Suppose M is a prime Γ-ring and $f: M \to M$ be an additive mapping. If f is centralizing on a left ideal J of M, then $f(a) \in Z(M)$, for all $a \in J \cup Z(M)$.

Proof: f is centralizing on left ideal J of M, we have
$[f(a), a]_\alpha \in Z(M)$ for all $a \in J$ and $\alpha \in \Gamma$.

By linearization, we have

$a, b \in J \Longrightarrow a + b \in J$, for all $\alpha \in \Gamma$.

$[f(a+b), a+b]_\alpha \in Z(M)$

f is an additive mapping then

$[f(a) + f(b), a+b]_\alpha \in Z(M)$

$[f(a), a]_\alpha + [f(a), b]_\alpha + [f(b), a]_\alpha + [f(b), b]_\alpha \in Z(M)$

f is a centralizing on left ideal J of M then, we get
$[f(a), a]_\alpha = 0$, $[f(b), b]_\alpha = 0$

$[f(a), b]_\alpha + [f(b), a]_\alpha \in Z(M)$, for all $a, b \in J$ and $\alpha \in \Gamma$. (1)

If $a \in Z(M)$, then equation (1) becomes
$[f(a), b]_\alpha \in Z(M)$.

Replacing b by $f(a)\beta b$ in above equation then, we get
$[f(a), f(a)\beta b]_\alpha \in Z(M)$

$[f(a), f(a)]_\alpha \beta b + f(a)\beta[f(a), b]_\alpha \in Z(M)$

$f(a)\beta[f(a), b]_\alpha \in Z(M)$. If $[f(a), b]_\alpha = 0$.

Then $f(a) \in C_{\Gamma M}(J)$.

The centralizer of J in M and hence$f(a) \in Z(M)$. Otherwise, if $[f(a), b]_\alpha \neq 0$, remark 1 follows that $f(a) \in Z(M)$. Hence the lemma.

Theorem 1: Let M be a prime Γ-ring satisfying the assumption (A) and D is a nonzero derivation on M. If f is a generalized Jordan derivation on a left ideal J of M such that f is commuting on J, then M is commutative.

Proof: Since f is commuting on J, we have
$[f(a), a]_\alpha = 0$, for all $a \in J$ and $\alpha \in \Gamma$.

Replacing a by $a + b$ in above equation, we get

$[f(a+b), a+b]_\alpha = 0$

$[f(a) + f(b), a+b]_\alpha = 0$

$[f(a), a]_\alpha + [f(a), b]_\alpha + [f(b), a]_\alpha + [f(b), b]_\alpha = 0$

$[f(a), b]_\alpha + [f(b), a]_\alpha = 0$ (2)

Replacing b by $a\beta a$ in equation (2), we get

$[f(a), a\beta a]_\alpha + [f(a\beta a), a]_\alpha = 0$

$[f(a), a]_\alpha \beta a + a\beta\, [f(a), a]_\alpha + [f(a)\beta a + a\beta D(a), a]_\alpha = 0$

$[f(a), a]_\alpha \beta a + a\beta\, [f(a), a]_\alpha + [f(a)\beta a, a]_\alpha + [a\beta D(a), a]_\alpha = 0$

$[f(a), a]_\alpha \beta a + a\beta\, [f(a), a]_\alpha + f(a)\beta [a, a]_\alpha + [f(a), a]_\alpha \beta a +$
$[a\beta D(a), a]_\alpha = 0$

f is centralizer, then $[f(a), a]_\alpha \beta a = 0, a\beta\, [f(a), a]_\alpha = 0,\ [f(a), a]_\alpha \beta a =$
$0, f(a)\beta[a, a]_\alpha = 0$.

$[a\beta D(a), a]_\alpha = 0$ (3)

Replacing $a\beta$ by $b\beta$ in equation (3), we get

$[b\beta D(a), a]_\alpha = 0$

Replacing b by $r\gamma a$ in above equation, then we get

$[r\gamma a\beta D(a), a]_\alpha = 0$

$r\gamma a\beta\, [D(a), a]_\alpha + [r\gamma a, a]_\alpha \beta D(a) = 0$

$r\gamma a\beta\, [D(a), a]_\alpha + r\gamma\, [a, a]_\alpha \beta D(a) + [r, a]_\alpha \gamma a\beta\, D(a) = 0$

$[r, a]_\alpha \gamma a\beta\, D(a) = 0$, for all $a \in J, r \in M$ and $\alpha, \beta, \gamma, \in \Gamma$.

Since M is prime Γ-ring, thus $[r, a]_\alpha = 0$ or $D(a) = 0$

Since D is nonzero derivation on M, then by lemma 2, D is nonzero on J.

Suppose $D(a) \neq 0$ for some $a \in J$, then $a \in Z(M)$.

Let $c \in J$ with $c \neq Z(M)$. Then $D(c) = 0$ and $a + c \notin Z(M)$, that is, $D(a + c) = 0$ and so $D(a) = 0$, which is a contradiction. Thus $c \in Z(M)$ for all $c \in J$. Hence J is commutative and lemma3, we get M is commutative.

Theorem 2: Let M be a prime Γ-ring satisfying the assumption (A) and J a left ideal of M with $J \cap Z(M) \neq 0$. If f is a generalized Jordan derivation on M with associated nonzero derivation D such that f is commuting on J, then M is commutative.

Proof: we claim that, $Z(M) \neq 0$ because of f is commuting on J and the proof is complete.

Now from equation (1), we get

$$[f(a), b]_\alpha + [f(b), a]_\alpha \in Z(M)$$

We replace a by $c\beta c$ with $0 \neq c \in Z(M)$, we get

$$[f(c\beta c), b]_\alpha + [f(b), c\beta c]_\alpha \in Z(M)$$

$$[f(c)\beta c + c\beta D(c), b]_\alpha + [f(b), c]_\alpha \beta c + c\beta[f(b), c]_\alpha \in Z(M)$$

$$[f(c)\beta c, b]_\alpha + [c\beta D(c), b]_\alpha + [f(b), c]_\alpha \beta c + c\beta[f(b), c]_\alpha \in Z(M)$$

$$f(c)\beta[c, b]_\alpha + [f(c), b]_\alpha \beta c + c\beta[D(c), b]_\alpha + [c, b]_\alpha \beta D(c) + [f(b), c]_\alpha \beta c + c\beta\, [f(b), c]_\alpha$$

$$\in Z(M)$$

$c \in Z(M) \Longrightarrow [c, b]_\alpha = 0$, for all $b \in J$.

Since $c \in Z(M) \Longrightarrow f$ is a centralizer on J.

$f(b) \in Z(M) \Longrightarrow [f(b), c]_\alpha = 0.$

$$[f(c), b]_\alpha \beta c + c\beta[D(c), b]_\alpha \in Z(M)$$

From lemma 1, $f(c) \in Z(M)$ and hence $c\beta[D(c), b]_\alpha \in Z(M)$.

Replacing b by $b + c$ in above equation, we get

$c\beta[D(c), b + c]_\alpha \in Z(M)$.

$c\beta[D(c), b]_\alpha + c\beta[D(c), c]_\alpha \in Z(M)$.

And consequently $c\beta[D(c), c]_\alpha \in Z(M)$.

As c is nonzero, remark 1 follows that $[D(c), c]_\alpha \in Z(M)$. This implies D is centralizing on J and hence we conclude that M is commutative.

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